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M. Sc. Mathematics 4th Semester (CBCS Scheme) Examination – May, 2024

ALGEBRAIC NUMBER THEORY

Paper : 17MAT24DB1

Time : Three Hours] *[Maximum Marks : 80*

Before answering the questions, candidates should ensure that they have been supplied the correct and complete question paper. No complaint in this regard, will be entertained after examination.

Note : Attempt *five* questions in all, selecting *one* question from each Section. Question No. 9 (Section – V) is ***compulsory***. All questions carry equal marks.

SECTION – I

1. (a) Show that $\sqrt{-5}/3$ is an algebraic number but not an algebraic integer. 8

   (b) State and prove fundamental theorem of arithmetic in $z[i]$. 8

2. (a) Prove that the units of field $Q(\sqrt{2})$ are $\pm u^{th}$ $(n = 0, 1, \ldots\ldots)$, where $u = 1 + \sqrt{2}$. 8

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(b) Prove that the number of real Euclidean fields $K = Q(\sqrt{m})$ where $m \equiv 2$ or 3 (mod 4) is finite. 8

SECTION – II

3. (a) Show that the set of algebraic number over Q is denumerable set. 8

(b) Show that $\sum_{n=0}^{\infty} \frac{1}{10^{n!}}$ is transcendental. 8

4. (a) Show that each Liouville number is a transcendental number. 8

(b) State and prove Liouville theorem of primitive elements. 8

SECTION – III

5. (a) Prove that the belinear pairing $B(x, y) : K \times K \to Q$ such that $B(x, y) = Tr_{K|Q}(xy)$ is a non-degenerate. 8

(b) Show that there always exists $w_1^*, \ldots\ldots, w_n^*$ in K such that $O_K \subseteq Zw_1^* + Zw_2^* + \ldots + Zw_n^*$. 8

6. (a) Let K be an algebraic number of degree n over Q, then show that O_K has an integral basis. 8

(b) Let $d(\neq 0, 1)$ be square free and $K = Q(\sqrt{d})$. If $d \equiv 1$ (mod 4), then prove that $1, \frac{1+\sqrt{d}}{2}$ is an integral basis for O_K and $O_K = Z = \left[\frac{1+\sqrt{d}}{2}\right]$. 8

SECTION – IV

7. (a) If $A \subseteq B$ are ideals of O_K, then show that $[O_K : A] > [O_K : B]$. 8

   (b) Show that sum and product of two fractional ideals are again fractional ideal. 8

8. (a) State and prove unique factorization theorem for ideals. 8

   (b) State and prove Chinese Remainder theorem. 8

SECTION – V

9. (a) Define norm of a Gaussian integer. 16

   (b) Obtain units in $Q(\sqrt{2})$.

   (c) State Liouville theorem.

   (d) Define algebraic number field.

   (e) What is sign of the discriminant ?

   (f) Define trace of an algebraic number.

   (g) Define integral closure.

   (h) Define ramified extensions.