76461

M. Sc. (Mathematics) 3rd Semester CBCS Scheme w.e.f. 2017-18 Examination – November, 2023

FUNCTIONAL ANALYSIS

Paper : 17MAT23C1

Time : Three Hours] *[Maximum Marks : 80*

Before answering the questions, candidates should ensure that they have been supplied the correct and complete question paper. No complaint in this regard, will be entertained after examination.

Note : Attempt ***five*** questions in all, selecting ***one*** question from each Section. Question No. 9 (Section-V) is ***compulsory***. All questions carry equal marks.

SECTION – I

1. (a) Prove that every normed linear space N is a metric space w.r.t. the metric defined by $d(x, y) = ||x - y||$. But the converse is not true.

(b) Suppose a Banach space B is directed sum of the linear subspaces M and N, i.e., $B = M \oplus N$ and let $z \in B$ has unique expression $z = x + y$, as the sum of x in M and y in N. Denote $||z||' = ||x + y||' =$

$||x|| + ||y||$. Check it is actually a norm. Let B' denote the linear space equipped with this norm. Prove that B' is a Banach space if M and N are closed in B.

2. (a) Let p be a real number such that $1 \leq p < \infty$. We denote by l_p^n, the space of all n-tuples & $x = (x_1, x_2, \ldots\ldots\ldots, x_n)$ of scalars with the norm defined by $P\, x\, P_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, then we have to prove that l_p^n is Banach space.

(b) Prove that for $1 \leq p \leq \infty$, L^p is complete.

SECTION – II

3. (a) Let N and N' be normed linear spaces and T is a linear transformation from N into N', then T is continuous either at every point of N or no point of N.

(b) Let N and N' be normed linear space over same scalar field. Then prove that a linear transformation $N \to N'$ is bounded if and only if it is continuous.

4. (a) Let N and N' be normed linear spaces, then N and N' are topologically isomorphic iff there exists a linear transformation T of N onto N' and positive constants m and M such that $m\, ||x|| \leq ||Tx|| \leq M\, ||x||\ \forall\, x \in N$.

(b) Let N be a normed linear space and x_0 is a non-zero vector in N, then prove that there exists a functional F in N^* s.t. $F(x_0) = ||x_0||$ and $||F|| = 1$.

SECTION – III

5. State and prove Riesz-Representation Theorem for bounded linear functional on $C[a, b]$.

6. (a) State and prove uniform boundedness principle.

(b) Let T be a linear transformation from D to normed linear space Y, where D is a subspace of X. Then if Y is a Banach space and T is closed and bounded. Then D is a closed subspace of X.

SECTION – IV

7. Prove that in a finite dimensional space X, all norms are equivalent.

8. (a) Let X and Y be Banach spaces. Then, $B(X, Y)$ is complete in strong sense. [$B(X, Y)$ is complete in strong sense, we mean that every Cauchy sequence converge to strongly to some element of $B(X, Y)$].

(b) Let X and Y be normed linear spaces and $T : X \to Y$ be a linear operator, then T is compact iff it maps every bounded sequence $< x_n >$ in X onto a sequence $< T(x_n) >$ in Y, which has a convergent subsequence.

SECTION – V

9. *Compulsory question :*

(i) Define Cauchy and Minkowski inequalities in R^n.

(ii) In a normed linear space, prove that every convergent sequence is a Cauchy sequence.

(iii) Write down equivalent *conditions to* T where T is a continuous linear transformation from $N \to N'$ & N and N' be normed linear space.

(iv) Define topological isomorphism and isometric isomorphism.

(v) Write down conjugate spaces of l_p and l_∞^n.

(vi) Give applications of closed graph theorem.

(vii) Prove that if a normed linear space X is reflexive, then it is weakly complete.

(viii) State F. Riesz Theorem.