M.Sc. 3rd Semester (DDE) Examination,

December–2022

MATHEMATICS

Paper–21MAT23C1

Functional Analysis

Time allowed : 3 hours] *[Maximum marks : 80*

Note : *Attempt five questions in all selecting **one** question from each unit.*

Unit–I

1. (a) Let $(X, \|\cdot\|)$ be a normed linear space. Show that the following assertions are equivalent 8

   (i) X is a Banach space

   (ii) Every absolutely convergent series in X converges.

   (b) Let p be a real number such that $1 \le p < \infty$. Denote by ρ^n_p the space of all n-tuples $x = <x_1, x_2, \ldots x_n>$ of scalars. Show that ρ^n_p is a Banach space under the norm 1/p. $\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}}$. 8

2. (a) Let N be a normed linear space over a field (C or R). Then prove that the mapping : $f : N \times N \to f(x, y) = x + y$ and $g : F \times N \to f(\alpha, x) = \alpha x$ are continuous. 8

   (b) Prove that the linear space ℓ^∞ of all bounded scalar sequences with the sup norm is a Banach space. 8

Unit–II

3. (a) Let N and N' be normed linear spaces and T be a continuous linear transformation of N into N' If M is the null space of T, then prove that T induces a natural linear transformation T' of N / M into N' are that $\|T'\| = \|T\|$. 8

   (b) Let N and N' be normed linear spaces and T be a linear transformation of N' into N. Show that the inverse T^{-1} exists and is continuous on its domain of definition if and only if there exists a constant $m > 0$ such that $m \|x\| \le \|T(x)\| \ \forall \ x \in N$. 8

4. (a) If T is bounded linear operator such that its inverse T^{-1} exists prove that T^{-1} is also continuous. 8

(b) Let N be a normed linear space over the field k. If T, S $\in$ B (N, N), then show that ST $\in$ B (N, N) and $\|ST\| \leq \|S\| . \|T\|$. 8

Unit–III

5. (a) Define reflexive space. Prove that ℓ_p (p > 1) is reflexive. 8

(b) Show that if a normed space is reflexive then it is necessarily a Banach space. Give an example to show that the converse is not true in general. 8

6. (a) Let X and Y be complete normed linear spaces and let T be a linear transformation of X into Y. Show that T is continuous if and only if its graph is closed in X × Y. 8

(b) State and prove the open mapping theorem. 8

Unit–IV

7. (a) Let $< x_n >$ be a sequence in a normed space X. Then show that

(i) Strong convergence implies weak convergence with the same limit.

(ii) The converse of (a) is not generally true.

(iii) If dim X < ∞, then weak convergence implies strong convergence. 8

(b) Prove that in a finite dimensional space, all norms are equivalent. 8

8. (a) Show that every compact linear operator is continuous. Also give example to show that converse of this is not true. 8

(b) Show that T : $\ell^2 \to \ell^2$ defined by $Tx = y = (\eta_j)$; $x_j = \frac{\ell_{uj}}{2^j}$ is compact. 8

Section–V

9. (a) State Uniform boundedness principle. 16

(b) Give example of a non-reflexive space.

(c) State Hahn-Banach extension theorem.

(d) Define boundedness and norm of a linear transformation.

(e) If X and Y are any two elements in a normed space X, then show that

$$|\|x\| - \|y\|| \leq \|x - y\|$$

(f) Define equivalent norms on a normed space. Also give suitable example.

(g) Show that sum of two compact linear operators is again compact. Define equivalent norms.

(h) Define Norms.