Roll No. ....

76470

M. Sc. (Mathematics) 3rd Semester
CBCS Scheme w.e.f. 2017-18
Examination – November, 2023

ANALYTICAL NUMBER THEORY

Paper : 17MAT23DB1

Time : Three Hours] *[Maximum Marks : 80*

Before answering the questions, candidates should ensure that they have been supplied the correct and complete question paper. No complaint in this regard, will be entertained after examination.

Note : Attempt ***five*** questions in all, selecting ***one*** question from each Section. Question No. 9 **(Section-V)** is ***compulsory***. All questions carry equal marks.

SECTION – I

1. (a) Prove that there are infinitely many primes of the form $8q + 5$. **8**

   (b) Let $a \geq 2$, $n \geq 2$ be natural numbers. Let $a^n - 1$ be a prime, then $a = 2$ and $n = p$ for some prime number p. **8**

2. (a) Let α be any rational number. Then there exists only a finite no. of pairs of integers x, y such that $x > 0, (x, y) = 1$ and $\left|\alpha - \frac{y}{x}\right| < \frac{1}{x^2}$. 8

(b) If x and y are positive integers then the following *two* inequalities $\frac{1}{xy} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{x^2} + \frac{1}{y^2}\right)$

$$\frac{1}{x(x+y)} \geq \frac{1}{\sqrt{5}}\left(\frac{1}{x^2} + \frac{1}{(x+y)^2}\right)$$

cannot hold simultaneously. 8

SECTION – II

3. The group U_n is cyclic iff $n = 1, 2, 4, p^r$ or $2p^r$, where p is an odd prime. 16

4. (a) Show that 2 is a quadratic residue of primes of the form $8k \pm 1$ and quadratic non-residue of primes of the form $8k \pm 3$. 8

(b) State and prove Quadratic Law of Reciprocity. 8

SECTION – III

5. Prove that every natural number n can be written as a sum of four squares. 16

6. (a) Show that the infinite series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for all real $s > 1$ and diverges for all real $s \geq 1$. 8

(b) Prove that $\zeta(2) = \frac{\pi^2}{6}$. 8

SECTION – IV

7. (a) Prove that followings : 8

(i) If $n = 2^{p-1}(2^p - 1)$ where p and $2^p - 1$ are both prime then n is perfect.

(ii) If n is even and perfect then n has the form given in (i).

(b) Prove that $\sigma(n)$ is a multiplicative function of n. Also find a formula for $\sigma(n)$. 8

8. (a) Prove that $\sum_{d/n} \frac{\mu(d)}{d} = \frac{\phi(n)}{n}$. 8

(b) Prove that average order of $\phi(n)$ is $\frac{6n}{\pi^2}$. In fact,

$\Phi(n) = \phi(1) + \phi(2) + ... + \phi(n) = \frac{3n^2}{\pi^2} + 0\,(n(\log n))$. 8

P. T. O.

SECTION – V

9. (a) Define Arithmetic function with example.

(b) State Mobius Inversion formula.

(c) Define primitive solution.

(d) Define Euler product.

(e) Define Mersenne prime with example.

(f) State Euclid's theorem.

(g) Show that $a = 2$ is a primitive root of U_{11}.

(h) Find the elements of Q_{25}. $2 \times 8 = 16$