M. Sc. 3rd Semester (DDE) Examination, December–2022

MATHS

Paper–21MAT23DB1

Analytical Number Theory

Time allowed : 3 hours] [*Maximum marks : 80*

Note : *Attempt five questions in all.* ***Question No. 9 from Section-V is compulsory.***

Section–I

1. (a) If $\alpha \geq 2, n \geq 2$ and $a^n + 1$ is a prime, then we must have $n = 2^k, k \geq 1$ and a is even.

   (b) Prove that if $\frac{h}{k}, \frac{h'}{k'}$ are two consecutive members of F_n, then

   (i) $k + k' > n$

   (ii) $k \neq k'$ if $n > 1$.

2. (a) Prove that e^y is irrational for all rationals $y \neq 0$.

   (b) Prove that if α is any rational number, then there exists only a finite number of pairs of integers x, y such that $x > 0, (x, y) = 1$ and $\left|\alpha - \frac{y}{x}\right| < \frac{1}{x^2}$.

Section–II

3. (a) Prove that an element $\alpha \in U_n$ is a primitive root if and only if $a^{\frac{\phi(n)}{q}} \neq 1$ in U_n for each prime q dividing ϕ (n).

   (b) Prove that if p is an odd prime then U_{p^e} is cyclic for $e \geq 1$.

4. (a) Prove that if $e \geq 3$ then $U_{2^e} = \{\mp 5^i \mid 0 \leq i < 2^{e-2}\}$.

   (b) Let a be an odd integer. Then, prove that

   (i) $a \in Q_2$;

   (ii) $a \in Q_4$ if and only if $a \equiv 1 \pmod 4$;

   (iii) if $e \geq 3$, then $a \in Q_{2^e}$ if and only if $a \equiv 1 \pmod 8$.

Section–III

5. (a) Derive the representation of Riemann zeta function as Euler's product.

   (b) If f is multiplicative and $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent then prove that

   $$\sum_{n=1}^{\infty} f(n) = \prod_{p} (1 + f(p) + f(p^2) + \ldots\ldots) \text{ where } p$$

   ranges over all primes.

6. (a) Prove that $x^4 + y^4 = u^2$ has no *non-trivial* solution.

(b) Prove that $g(k) \geq \left[\left(\frac{3}{2}\right)^4\right] + 2^k - 2$.

Section–IV

7. (a) Prove that if n is any natural number then

$$\sum_{d/n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

(b) Prove that $\phi(n)$ is a multiplicative function.

8. (a) Prove that

(i) $\sum_{d/n} \tau(d)\mu\left(\frac{n}{d}\right) = \sum_{d/n} \mu(d)\tau\left(\frac{n}{d}\right) = 1$ and

(ii) $\sum_{d/n} \sigma(d)\mu\left(\frac{n}{d}\right) = \sum_{d/n} \mu(d)\sigma\left(\frac{n}{d}\right) = n$

(b) Prove that d(n) is of average order of log n.

Section–V

9. **Compulsory Quesitou :**

(i) Prove that $p_{n+1} \leq 2^{2^n} + 1 = F_n$ where p_i denotes ith prime in ascending order.

(ii) State and prove Pythagoras Theorem.

(iii) Prove that Q_n is a subgroup of U_n.

(iv) Define primitive Roots with example.

(v) Evaluate $\zeta(2)$ where is Riemann zeta function.

(vi) Explain Waring problem.

(vii) Prove that if $2^{n+1} - 1$ is a prime number, then $m = 2^n (2^{n+1} - 1)$ is perfect.

(viii) Prove that if p is a prime such that p and $p + 2$ are both primes then $\phi(p + 2) = \phi(p) + 2$.