

M.Sc. 2nd Semester CBCS Scheme w.e.f. 2016-17  
Examination, July-2022

MATHS

Paper-16MAT22C2

Measure and Integration Theory

*Time allowed : 3 hours ] [ Maximum marks : 80*

*Note : Attempt five questions in all, selecting one question from each unit. Question No. 9 is compulsory*

**Unit-I**

1. (a) Prove that outer measure of an interval is equal to its length.  
(b) Show that union of two measurable sets is measurable.
2. (a) Let  $E$  be a given set. Then prove that the following statements are equivalent :
  - (i)  $E$  is measurable.
  - (ii) Given  $\epsilon > 0$ , there is an open set  $O \supset E$  such that  $m^*(O - E) < \epsilon$ .
  - (iii) There is a  $G_\delta$  - set  $G \supset E$  such that  $m^*(G - E) = 0$ .

(2)

- (iv) Given  $\epsilon > 0$ , there is a closed set  $F \subset E$  such that  $m^*(E - F) < \epsilon$ .
- (v) There is a  $F_\sigma$  - set  $F \subset E$  such that  $m^*(E - F) = 0$ .
- (b) Let  $E \subset [0, 1)$  be a measurable set and  $y \in [0, 1)$  be given. Then prove that the set  $E + y$  is measurable and  $m(E + y) = m(E)$ .

**Unit-II**

3. (a) Prove that a continuous function defined on a measurable set is a measurable function.  
(b) State and Prove Approximation theorem for simple functions.
4. (a) Let  $F$  be a closed subset of  $\mathbb{R}$ , then prove that a function  $g : F \rightarrow \mathbb{R}$  is continuous if sets  $\{x : g(x) \leq a\}$  and  $\{x : g(x) \geq b\}$  are closed subsets of  $F$  for every rational  $a$  and  $b$ .  
(b) Let  $\{f_n\}$  be a sequence of measurable function which converges to  $f$  a. e. on  $X$ . Prove that  $f_n$  is convergence in measure to  $f$  on  $X$ .

## Unit-III

5. (a) Compare Lebesgue and Riemann Integration.
- (b) If  $f = g$  a. e. then show that  $\int_E f = \int_E g$  and converse is not true.
6. (a) Let  $f$  and  $g$  be non-negative measurable functions on  $E$ . Then prove that for any  $\alpha > 0$ ,  $\beta > 0$ ,  $\int_I (\alpha f + \beta g) = \alpha \int_I f + \beta \int_I g$ .  
Moreover, if  $f \leq g$  on  $E$  then  $\int_E f \leq \int_E g$ .
- (b) Let  $f$  be a non-negative function which is integrable over a set  $E$ . Prove that given  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for every set  $A \subset E$  with  $m(A) < \delta$ , we have  $\int_A f < \epsilon$ .

## Unit-IV

7. (a) If  $f$  is monotonic function on  $[a, b]$ , then show that it is of bounded variation with total variation  $V_a^b(f) = |f(b) - f(a)|$ .
- (b) If  $f$  and  $g$  be functions of bounded variation then show that their product is also a function of bounded variation.

8. (a) If  $f$  is integrable on  $[a, b]$ , then prove that the function  $F$  defined by  $F(x) = \int_a^x f(t) dt$  is a continuous function of bounded variation on  $[a, b]$ .
- (b) If  $f$  is absolutely continuous function on  $[a, b]$ , then prove that it is function of bounded variation.

## Unit-V

9. (a) Show that outer measure of a null set is zero.
- (b) Define Boolean algebra of measurable sets.
- (c) Show that every step function is measurable.
- (d) Define limit superior and limit inferior of sequence of functions.
- (e) Define Lebesgue integral as a integration of simple function.
- (f) If  $m(E) = 0$  then show that  $\int_E f = 0$ .
- (g) Define function of bounded variation.
- (h) State Fundamental theorem of Integral Calculus.